Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra II

Mid-term Examination — Instructor: B V Rajarama Bhat
Total Marks 7x15=105 — Date : Feb. 25, 2026
Maximum marks: 100 — Time: 3 hours

Here $M_n(\mathbb{R}), M_n(\mathbb{C})$ denote $n \times n$ real and complex matrices respectively. The standard inner product is considered unless specified otherwise.

(1) Fix $n \in \mathbb{N}$ and take $S = \{1, 2, \ldots, n\}$. For any permutation σ of S, let $\epsilon(\sigma)$ denote the signature of σ. Show that $\epsilon(\tau \circ \sigma) = \epsilon(\tau).\epsilon(\sigma)$ for any two permutations σ, τ of S. [15]

(2) Let $B \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Show that

$$\det(B^*) = \overline{\det(B)}.$$

[15]

(3) Let $A \in M_n(\mathbb{R})$. Let u, v be two vectors in $\mathbb{R}^n$ and let $d \in \mathbb{R}$ with $d \neq 0$. Let $P \in M_{n+1}(\mathbb{R})$ be defined by

$$P = \begin{bmatrix} A & u \\ v' & d \end{bmatrix}.$$

(Here v' is the row vector got by taking the transpose of v.). Show that

$$\det(P) = d \det(A - \frac{1}{d}uv').$$

[15]

(4) State and prove Cauchy-Schwarz inequality (including the condition for equality) for inner product spaces. [15]

(5) Fix $d \geq 2$ and let $\mathcal{V}$ be an inner product space over $\mathbb{C}$ with an ortho-normal basis $\mathcal{B} = \{v_1, \ldots, v_d\}$. Suppose U is a linear map on $\mathcal{V}$ satisfying

$$Uv_j = \begin{cases} \omega^j v_{j+1} & \text{if } 1 \leq j < d; \\ v_1 & \text{if } j = d. \end{cases}$$

where $\omega = e^{\frac{2\pi i}{d}}$. (So ω is a d-th root of unity.) (i) Show that U is a unitary. (ii) Compute the determinant of U. (Hint: First write down the matrix of U.) [15]

(6) Let

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

considered as a linear map on $\mathbb{C}^3$. (i) Compute the matrix of the projection onto the range of D. (ii) Compute the matrix of the projection onto the kernel of D. [15]

(7) Fix $n \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_n$ be some real numbers. Define $A \in M_n(\mathbb{R})$ by $A = [a_{ij}]_{1 \leq i,j \leq n}$ where

$$a_{ij} = \delta_{ij} + a_i + a_j, \ 1 \leq i, j \leq n.$$

($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.). Compute the characteristic polynomial of A. [Caution: The value of n matters in these computations.] [15]